SETS OF UNIQUENESS AND SETS OF MULTIPLICITY(1)

BY

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ABSTRACT

This paper establishes a condition of metric thinness for a translation set E which suffices to imply that E is a set of uniqueness (in the broad sense). An existence proof is given to show that this result is close to being sharp. These theorems extend results of Raphaël Salem.

1. **Introduction.** Let the circle group T be identified with the real numbers modulo 1, by the correspondence $t \to e(t) = e^{2\pi i t}$. The distribution on T whose Fourier series is $\sum_{n=-\infty}^{\infty} c_n e(nx)$ is called a *pseudofunction* if $\lim_{|n|\to\infty} c_n = 0$. Let PF denote the class of such distributions. A closed set $E \subset T$ is called a set of *multiplicity*, or an $\mathscr{M}$*-set*, if it supports a nonzero pseudofunction; if it does not, it is a set of *uniqueness*, or a $\mathscr{U}$*-set*. If $M(E)$, the class of measures supported by E, contains a nonzero pseudofunction, then E is a set of *multiplicity in the strict sense*, or an $\mathscr{M}_0$*-set*; otherwise, a set of *uniqueness in the broad sense*, or a $\mathscr{U}_0$*-set*. (For a proof that not every $\mathscr{M}$-set is an $\mathscr{M}_0$-set, see [6], sections 1 and 3.)

Every set of positive measure is an $\mathscr{M}$-set. But both $\mathscr{U}$-sets and $\mathscr{M}$-sets occur among sets of zero measure, and the study of their properties has been a subject of some interest. Accounts of the work that has been done may be found in chapter IX of Zygmund's work [9] and Chapters V and VI of the book by Kahane and Salem [5].

It is natural to consider criteria of metric thinness which distinguish among sets of Lebesgue measure zero, namely their Hausdorff measure with respect to convex functions (see the end of this section for a definition). One might hope to find that whenever a set has zero Hausdorff measure with respect to a certain function h, it must be a $\mathscr{U}$-set. But by itself such a hypothesis does not suffice; Ivačev-Musatov [3,4] has shown that for an arbitrary convex function h, there is an $\mathscr{M}_0$-set of zero h-measure; and he provides an explicit method of construction, which we shall describe briefly later.

We shall show that if a translation set has zero h-measure, where $h(t) = (\log t^{-1})^{-1}$, then it is a $\mathscr{U}_0$-set. This result is a corollary of Theorem A.

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To state this theorem and discuss our results suitably, we need some technical conventions for describing translation sets (cf. [5], Chapter I).

Let $\boldsymbol{u} = (u_1, \cdots, u_H)$ with $H \geqq 2$ and

$$0 \leqq u_1 < u_2 < \cdots < u_H < 1;$$

and let ξ, called a *ratio of dissection*, satisfy

$$0 < \xi < \min\{1 - u_H,\ u_H - u_{H-1}, \cdots, u_2 - u_1\}.$$

A *dissection of type* $(H, \boldsymbol{u}, \xi)$ on an interval $E_0 = [a,b]$ is the selection of a set $E_1 \subset E_0$ consisting of the union of the H pairwise disjoint closed intervals with left-hand endpoints $a + (b-a)u_j$ and length $(b-a)\xi$. We shall say that we *keep* the intervals that make up E_1.

When a dissection of type $(H(1), \boldsymbol{u}_1, \xi_1)$ is made on the interval $[0,1]$, $J(1) = H(1)$ intervals of length $d_1 = \xi_1$ are kept; when a dissection of type $(H(2), \boldsymbol{u}_2, \xi_2)$ is made on each of these intervals, $J(2) = H(1)\ H(2)$ intervals of length $d_2 = \xi_1 \xi_2$ are kept. Repeating the procedure, after the kth dissection $J(k) = H(1) \cdots H(k)$ intervals are kept, each of length $d_k = \xi_1 \cdots \xi_k$, and we call their union E_k. We define the set $E = E\{(H(k), \boldsymbol{u}_k, \xi_k)\}$ to be the intersection $\bigcap_{k=1}^{\infty} E_k$. Its measure is $\prod_{k=1}^{\infty} H(k)\xi_k$. Write $\boldsymbol{u}_k$ as $(u_{k,1}, \cdots, u_{k,H(k)})$. Then the points of E are precisely the sums of all the infinite series

$$u_{1,j(1)} + d_1 u_{2,j(2)} + \cdots + d_{k-1} u_{k,j(k)} + \cdots \tag{1–1}$$

where $j(k) = 1, 2, \cdots,$ or $H(k)$ for each k.

A *portion* of a set is its non-empty intersection with an open interval. A perfect set which is decomposable, for an infinite number of integers H, into a union of H pairwise disjoint portions which are translates one of another, is called a *translation* set. A closed perfect set is a translation set if and only if it may be described by a construction of the sort just explained.

Actually, a translation set always may be described with

$$u_{k,1} = 0 \text{ for all } k. \tag{1–2}$$

We did not adopt (1–2) as a convention because in Section 3 it is advantageous to treat all the $u_{k,j}$ as variables. If (1–2) holds, and $H(k) \equiv 2$, and we let $r_k = u_{k,2} d_{k-1}$, then the sums (1–1) are precisely the sums

$$\sum_{k=1}^{\infty} \varepsilon_k r_k : \varepsilon_k = 0 \text{ or } 1 \text{ for } k = 1, 2, \cdots.$$

Such a set, or any translate thereof, is called a *symmetric* set and its description may be further conventionalized so that $\boldsymbol{u}_k = (0, 1 - \xi_k)$. We denote this set by $E\{\xi_k\}$. Among the symmetric sets with constant ratio of dissection, which we denote by $E(\xi)$, is the familiar Cantor set $E(1/3)$.

THEOREM A. *If $b > 4$ and $d_r = \xi_1 \cdots \xi_r$, and*

$$\liminf_{r \to \infty} d_r \exp[(\log br) \sum_{k=1}^{r} (H(k) - 1)] = 0, \tag{1–3}$$

then the translation set $E = E\{(H(k), u_k, \xi_k)\}$ is a $\mathscr{U}_0$-set.

This theorem will be proved, and its corollaries derived, in Section 2. It shows that there is a large class of translation sets in which the $\mathscr{U}_0$ property is stable under small changes in the parameters of construction.

Theorem A is close to being a sharp result, for if $\beta < \frac{1}{2}$, there exists a translation set which is an $\mathscr{M}_0$-set and for which

$$\sum_{r=1}^{\infty} d_r^{-1} \exp[-\beta(\log r) \sum_{k=1}^{r} (H(k) - 1)] < \infty. \tag{1–4}$$

This result was first obtained by Salem ([5], p. 100). It also follows from the somewhat technical result, Theorem C, which we shall state and prove in Section 3. The procedure is an extension of Salem's method, and yields, in particular, the following new result: There exists a translation set which is an $\mathscr{M}_0$-set and has zero f_b-measure for all the functions

$$f_b(t) = \exp[-(\log t^{-1})^b], \quad 0 < b < 1.$$

DEFINITION OF HAUSDORFF MEASURE (cf. [5], ch. II). Let $h(t)$ be a function defined for $t \in [0,1)$ with $h(0) = 0$, $h'(t) > 0$, $h''(t) \leqq 0$; for example, $h_a(t) = t^a$ for $0 \leqq a \leqq 1$. For a set E and $\varepsilon > 0$, let $C(\varepsilon, E)$ be the infimum of the sums $\sum_{j=1}^{\infty} h(|E_j|)$, where E is covered by a union of open intervals E_j, $j = 1, 2, \cdots$, with length $|E_j| \leqq \varepsilon$ for every j; then $C(\varepsilon, E)$ is nondecreasing as $\varepsilon \to 0$. The *Hausdorff measure of E with respect to h*, or the *h-measure* of E, is

$$\mu_h(E) = \lim_{\varepsilon \to 0} C(\varepsilon, E),$$

which may be infinite. Note that if E has zero g-measure, and if $h(t) = o(g(t))$ as $t \to 0$ (we say that g is *steeper* than h), then E also has zero h-measure.

2. Conditions which imply that a set is a $\mathscr{U}_0$-set.

Proof of Theorem A. To show that $M(E) \cap PF = \{0\}$, it suffices to show that

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| > 0 \tag{2–1}$$

for an arbitrary *positive* measure μ carried by E; for if a measure is a pseudofunction, so is its total variation. The latter fact follows from the existence of the Radon-Nikodym derivative of $|\mu|$ with respect to μ ([2], III.10.7); and the fact that if

$\lim_{|n|\to\infty}\int_0^1 e(nx)\mu(dx)=0$ and if f is μ-integrable, then $\lim_{|n|\to\infty}\int_0^1 e(nx)f(x)\mu(dx)=0$ ([7], p. 38, lemma, or [9], XII. 10.9).

We shall use the elementary result of Diophantine approximation theory ([1], p. 13) that, given K real numbers $t_1,\cdots,t_K$ and $\varepsilon>0$, there is an integer m such that

$$1\leqq|m|\leqq\varepsilon^{-K},\text{ and }\|mt_j\|<\varepsilon\text{ for }j=1,\cdots,K$$

(where $\|x\|$ denotes the distance from the real number x to the nearest integer).

We shall also use the inequalities

$$|e(x)-e(y)|=2\sin(\pi\|x-y\|)\leqq 2\pi\|x-y\|;$$

$$\|nx\|\leqq|n|\ \|x\|\quad(x,\ y\text{ real}).$$

Let $\mu\in M(E)$ be a positive measure of total mass $\|\mu\|_M=1$. The translation set E is, for each $r=1,2,\cdots$, covered by $J(r)$ intervals I_i of length $|I_i|\leqq d_r$; let $s_i\in I_i$. We have

$$\Big|\hat{\mu}(n)-\sum_{i=1}^{J(r)}\mu(I_i)\,e(-ns_i)\Big|$$
$$=\Big|\sum_{i=1}^{J(r)}\int_{I_i}(e(-nx)-e(-ns_i))\mu(dx)\Big|\leqq 2\pi|n|d_r.$$

For an integer n such that $\|ns_i\|\leqq 1/4,\ i=1,\cdots,J(r)$,

$$\Big|\sum_{i=1}^{J(r)}\mu(I_i)e(-ns_i)\Big|\geqq\sum_{i=1}^{J(r)}\mu(I_i)\cos 2\pi ns_i$$
$$\geqq\min_{1\leqq i\leqq J(r)}(\cos 2\pi\|ns_i\|);$$

because $\mu(I_i)\geqq 0$, $\sum_{i=1}^{J(r)}\mu(I_i)=1$, and $\cos 2\pi ns_i\geqq 0$. Therefore, for such n,

$$|\hat{\mu}(n)|\geqq\min_{1\leqq i\leqq J(r)}(\cos 2\pi\|ns_i\|)-2\pi|n|d_r$$

Thus to prove (2–1) it suffices to show that there exists a sequence of integers $\{n_r\}$ such that

$$\lim_{r\to\infty}|n_r|=\infty;\ \lim_{r\to\infty}|n_r|d_r=0;\quad\text{and}\tag{2–2}$$
$$\|n_rs_i\|<b^{-1}<\frac{1}{4}\ (i=1,\cdots,J(r)).$$

For then

$$\limsup_{|n|\to\infty}|\hat{\mu}(n)|\geqq\cos 2\pi b^{-1}>0.\tag{2–3}$$

First we shall carry out the proof assuming that

$$\lim_{r\to\infty} d_r \exp[(\log br) \sum_{k=1}^{r} (H(k)-1)] = 0. \tag{2–4}$$

Since E is a translation set, the s_i's (cf. (1–1)) may be taken to be the $J(r) = H(1)\cdots H(r)$ numbers

$$\{s_{j(1),\dots,j(r)} = \sum_{k=1}^{r} u_{k,j(k)} d_{k-1} : j(k) = 1,\cdots, H(k) \text{ for } k = 1,\cdots, r\}$$

— re-indexed as indicated. Therefore, in order to solve the $J(r)$ inequalities of (2–2) for n_r, it suffices to find n_r obeying the following $\sum_{k=1}^{r}(H(k)-1)$ inequalities:

$$\| n_r(u_{k,j(k)} d_{k-1}) \| < 1/br \ (j(k) = 2,\cdots, H(k);\ k = 1,\cdots, r)$$

— since it may be assumed without loss of generality that $u_{k,1} = 0$ for every k ((1–2)). By (2–4) it is possible to select a sequence of integers $\{m_r\}$ such that

$$\lim_{r\to\infty} m_r = \infty, \text{ but } \lim_{r\to\infty} d_r m_r \exp[(\log br) \sum_{k=1}^{r} (H(k)-1)] = 0.$$

For each r, there exists an m satisfying the $K = \sum_{k=1}^{r}(H(k)-1)$ inequalities

$$\| m(m_r u_{k,j(k)} d_{k-1}) \| < 1/br \ (j(k) = 2,\cdots, H(k);\ k = 1,\cdots, r);$$

and such that

$$1 \leqq |m| \leqq (1/br)^{-K} = \exp[(\log br) \sum_{k=1}^{r} (H(k)-1)].$$

With $n_r = mm_r$, the sequence $\{n_r\}$ satisfies the three conditions (2–2) and the theorem follows.

If we use (1–3) instead of (2–4), we proceed similarly. There is then a sequence $\{r(p): p = 1,2,\cdots\}$ of values of r for which there exist integers $n_{r(p)}$, such that (2–2) is satisfied if we write $r(p)$ for r and take the limits as $p\to\infty$. Then (2–3) still follows. The proof of the theorem is complete.

Applying the theorem to symmetric sets, for which $H(k) \equiv 2$, we obtain

COROLLARY A–1. *If $b > 4$ and*

$$\liminf_{r\to\infty} d_r \exp(r \log br) = 0$$

then the symmetric set $E\{\xi_k\}$ is a $\mathscr{U}_0$-set.

Conditions such as (1–3) have no simple relationship to Hausdorff measures. However, from Theorem A we can obtain

COROLLARY A-2. *If a translation set E has h-measure zero where*

$$h(t) = (\log t^{-1})^{-1},$$

then E is a $\mathscr{U}_0$-set.

Proof. Let $E = E\{(H(k), u_k, \xi_k)\}$. The hypothesis implies that

$$\liminf_{r\to\infty} J(r)h(d_r) = 0.$$

Therefore

$$\liminf_{r\to\infty}[\log J(r) + \log h(d_r)]$$

$$= \liminf_{r\to\infty}\left[\sum_{k=1}^{r} \log H(k) - \log\log d_r^{-1}\right] = -\infty;$$

therefore

$$\liminf_{r\to\infty}\left[\log\log(br) + \log\sum_{k=1}^{r} H(k) - \log\log d_r^{-1}\right] = -\infty;$$

therefore

$$\liminf_{r\to\infty}\left[(\log br)\sum_{k=1}^{r} H(k) - \log d_r^{-1}\right] = -\infty;$$

therefore

$$\liminf_{r\to\infty} d_r \exp\left[(\log br)\sum_{k=1}^{r} H(k)\right] = 0,$$

and by the theorem, E is a $\mathscr{U}_0$-set.

Now we shall present a result which in one sense is more general than Theorem A. But the hypothesis, like that of A, is of interest as a criterion of metric thinness only for sets with a certain regularity of construction. The theorem differs only slightly from some results due to Salem ([8], theorems VII, IX, and X; or [5] ch. VII, theorems XIV and XVI).

THEOREM B. *Let E be a closed perfect subset of the circle, $\{d_r\}$ a null sequence of positive numbers, $\{J(r)\}$ a sequence of integers; such that for each $r = 1, 2, \cdots$, the set E is covered by $J(r)$ intervals, each of length no greater than d_r. If for some $a > 0$,*

$$\liminf_{r\to\infty} d_r e^{aJ(r)} = 0, \tag{2-5}$$

then E is a $\mathscr{U}_0$-set.

Proof. If the theorem holds in the case $a = c$, then it holds in the case $a = c/2$, by the following argument. Let $\mu \in M(E) \cap PF$ and suppose (2–5) holds for E with $a = c/2$. Since E is a closed perfect subset of the circle, it is the union of two closed perfect sets E_1, E_2 contained respectively in two disjoint intervals, each of length less than $2/3$. Then (2–5) holds, with $a = c$, for at least one of the two sets, say for E_1, so that E_1 is a $\mathscr{U}_0$-set. Since μ is a pseudofunction, the restriction of μ to E_1 is a pseudofunction (cf. [7], p. 38, lemma) and hence must be the zero measure; so E_2 must support μ, and (2–5) holds for E_2 with $a = c/2$. Repeating the argument, we show that the support of μ is contained in an interval of length $\lim_{k\to\infty}(2/3)^k = 0$ and hence has a single point for its support. Since μ is a pseudofunction, $\mu = 0$. Thus the theorem holds in the case $a = c/2$.

Therefore to prove the theorem it suffices to prove it for the case of an arbitrary $a = b > \log 4$, say. The proof proceeds as for Theorem A, except that the s_i's cannot be chosen in so convenient a manner; for each r, the integer n_r must solve $J(r)$ instead of only $\sum_{k=1}^{r}(H(k)-1)$ inequalities. But the hypothesis (2–5) allows us to find a sequence of integers satisfying the conditions (2–2), proving the theorem.

REMARK. As we mentioned earlier, Ivačev-Musatov has shown that for every h there is an $\mathscr{M}_0$-set with zero h-measure. If h is sufficiently steep, of course, such a set cannot obey requirements such as (2–5) or (1–3).

What (2–5) and (1–3) say, about a set E, is that for a null sequence of numbers $d > 0$, it is possible to cover E with intervals of length d with a certain efficiency — that is, without using too many such intervals. The sets of Ivačev-Musatov are constructed in such a way that such efficient coverings are not possible. It is impractical to relate his procedure here, but let us give a brief indication: $E = \bigcap_{k=1}^{\infty} E_k$, $E_k = \bigcup_{j=1}^{J(k)} I_{kj}$, where the I_{kj} are of widely differing lengths. In determining E_{k+1}, a different dissection is performed on each of the intervals $I_{k1}, \cdots, I_{kJ(k)}$, and these dissections are progressively "more severe" as j counts from 1 to $J(k)$; the intervals of E_{k+1} placed in I_{k2} will be much smaller and more numerous than those placed in I_{k1}; etc. Thus E is far less regular in structure than, say, a translation set — necessarily so, according to our results.

3. An existence proof for thin translation sets of multiplicity.

We shall define a measure ν on a class $\mathscr{E}$ of translation sets. For each $E \in \mathscr{E}$ we shall specify, in a natural way, a certain $\mu \in M(E)$. If $c_n = \hat{\mu}(n)$, we shall show that

$$c_n \to 0 \text{ as } |n| \to \infty \tag{3–1}$$

a.e. in $(\mathscr{E}, \nu)$, which implies that almost all sets E in the measure space $(\mathscr{E}, \nu)$ are $\mathscr{M}_0$-sets.

If $\{q(n)\}$ is an increasing sequence of integers, the series $\sum_{n=0}^{\infty}|c\, c_n|^{2q(n)}$ converges for $|c| < (\limsup_{n\to\infty}|c_n|)^{-1}$. If for some sequence $\{q(n)\}$,

$$\sum_n |c\, c_n|^{2q(n)} < \infty \text{ a.e. in } (\mathscr{E}, \nu) \text{ for all } c > 0, \tag{3–2}$$

then we know that $\limsup_{n\to\infty}|c_n| < c^{-1}$ for all $c > 0$, for almost all $E \in \mathscr{E}$. Assuming that the same can be done for negative values of n, (3–1) follows. To prove (3–2), it suffices to show that

$$\sum_n \int_{(\mathscr{E},\nu)} |c\, c_n|^{2q(n)} < \infty \text{ if } c > 0. \tag{3–3}$$

We shall prove (3–3) for a certain sequence $\{q(n)\}$, and thus obtain (3–1).

Let $\{H(k), \xi_k \colon k = 1, 2, \cdots\}$ be fixed. We define $\mathscr{E}$ to be the class of translation sets $E = E\{(H(k), \boldsymbol{u}_k, \xi_k)\}$ for which

$$\left|u_{k,j} - \frac{3j-2}{3H(k)}\right| \leqq \frac{1}{3H(k)}, j = 1, \cdots, H(k); k = 1, 2, \cdots. \tag{3–4}$$

Recall that $E = \bigcap_{k=1}^{\infty} E_k$, where E_k is the union of $J(k)$ intervals, each of length d_k. Let $L_k(x)$ be the function on $[0,1]$ which is equal to zero at 0, increases linearly by the amount $J(k)^{-1}$ on each of the $J(k)$ intervals of E_k, and stays constant within each of the intervals contiguous to E_k in $[0,1]$; so that $L_k(1) = 1$. As $k \to \infty$, $\{L_k(x)\}$ converges uniformly to a continuous nondecreasing function $L(x)$ which is constant on each interval contiguous to E in $[0,1]$. The measure $\mu = dL$ thus has total mass 1, is carried by E, and is called the *Lebesgue measure on E*. Another way of defining μ is as an infinite convolution $\mu_1 * \mu_2 * \cdots$ of measures with finite support. Let μ_1 assign mass $H(1)^{-1}$ to each of the points $u_{1,j}$, $j = 1, \cdots, H(1)$. Let μ_2 assign mass $H(2)^{-1}$ to each of the points $(d_1 u_{2,j})$, $j = 1, \cdots, H(2)$. In general, let μ_k assign mass $H(k)^{-1}$ to each of the points $(d_{k-1} u_{k,j})$, $j = 1, \cdots, H(k)$. The nth Fourier-Stieltjes coefficient of μ_k is, letting $d_0 = 1$,

$$\hat{\mu}_k(n) = H(k)^{-1} \sum_{j=1}^{H(k)} e(-n\, d_{k-1} u_{k,j}).$$

The Fourier-Stieltjes coefficients of μ, which is the limit of

$$\mu_1 * \mu_2 * \cdots * \mu_k = dL_k,$$

are given by

$$c_n = \hat{\mu}(n) = \prod_{k=1}^{\infty} Q_k(-n d_{k-1}), \tag{3–5}$$

where

$$Q_k(w) = H(k)^{-1} \sum_{j=1}^{H(k)} e(u_{k,j} w). \tag{3–6}$$

Thus for each k, $Q_k(w)$ is a function of the variables $u_{k,j}$, $j = 1,\cdots,H(k)$, which we parametrize in turn as follows:

$$u_{k,j} = u_{k,j}(t_{k,j}) = \frac{3j-2}{3H(k)} + \frac{t_{k,j}}{3H(k)a_k} . \tag{3-7}$$

We allow *all real values* for each $t_{k,j}$. Of course, only when

$$\left|t_{k,j}\right| \leqq a_k,\ j = 1,\cdots,H(k), \tag{3-8}$$

does $\boldsymbol{u}_k = \boldsymbol{u}_k(\boldsymbol{t}_k)$ obey (3–4); only then will (3–6) and (3–5) have relevance to sets in $\mathscr{E}$. Let us now indicate why we consider all real $t_{k,j}$ and how we select the positive constants a_k.

We consider $Q_k(w)$ as a function defined on the product space

$$R^{H(k)} = \{\boldsymbol{t}_k = (t_{k,1},\cdots,t_{k,H(k)}) : t_{k,j} \in R \text{ for } j = 1,\cdots,H(k)\},$$

where R is the real line. Let m_k be the product measure obtained on $R^{H(k)}$by placing the measure $(2\pi)^{-1/2}\exp(-t^2/2)dt$, the normal distribution, on each coordinate R.

We now consider c_n, as given by (3–5), to be a function $c_n(\tau)$ defined on the product of all these measure spaces:

$$(\mathscr{T}, \nu) = \prod_{k=1}^{\infty} (R^{H(k)}, m_k);$$

$$\mathscr{T} = \{\tau = (\boldsymbol{t}_1, \boldsymbol{t}_2, \cdots) = (t_{1,1},\cdots,t_{1,H(1)}; t_{2,1},\cdots,t_{2,H(2)};\cdots)\}.$$

Let $\mathscr{T}_1$ be the set of $\tau \in \mathscr{T}$ such that (3–8) holds for all k. It has measure

$$\begin{aligned} \nu(\mathscr{T}_1) &= \prod_{k=1}^{\infty} \left[(2\pi)^{-1/2}\int_{-a_k}^{a_k}\exp(-t^2/2)dt\right]^{H(k)} \\ &\geqq \prod_{k=1}^{\infty} [1-\exp(-a_k^2/2)]^{H(k)} \qquad (3\text{–}9) \\ &\geqq \prod_{k=1}^{\infty} [1 - H(k)\exp(-a_k^2/2)]. \end{aligned}$$

If $\tau \in \mathscr{T}_1$, then $c_n = c_n(\tau)$ is the nth coefficient of the Lebesgue measure on the set $E = E(\tau) = E\{(H(k), \boldsymbol{u}_k(\boldsymbol{t}_k), \xi_k)\}$; and

$$\mathscr{E} = \{E(\tau) : \tau \in \mathscr{T}_1\}.$$

So we may regard the restriction of ν to $\mathscr{T}_1$ as a measure on the class $\mathscr{E}$. It will be a non-trivial measure only if $\nu(\mathscr{T}_1) > 0$, for which, by (3–9), it suffices to have

$$\prod_{k=1}^{\infty} [1 - H(k)\exp(-a_k^2/2)] > 0.$$

To insure this relation we adopt the convention that

$$a_k = 10\ (\max\{k, H(k)\})^{1/2}. \tag{3–10}$$

We shall show that for a certain increasing sequence of integers $\{q(n)\}$,

$$\sum_n \int_{(\mathscr{T},\nu)} \left| c\, c_n(\tau) \right|^{2q(n)} < \infty \text{ if } c > 0. \tag{3–11}$$

Hencein particular, (3–3) holds, so that the Lebesgue measure is a pseudofunction a.e. in $(\mathscr{E}, \nu)$.

For each n we shall select an integer $p(n)$ and use the fact that

$$\int_{(\mathscr{T},\nu)} \left| cc_n(\tau) \right|^{2q(n)} = \int_{(\mathscr{T},\nu)} \left| c \prod_{k=1}^{\infty} Q_k(-nd_{k-1}) \right|^{2q(n)}$$

$$\leqq \int_{(R^{H(1)},m_1)} \cdots \int_{(R^{H(p(n))},m_{p(n)})} \left| c \prod_{k=1}^{p(n)} Q_k(-nd_{k-1}) \right|^{2q(n)} \tag{3–12}$$

$$= c^{2q(n)} \prod_{k=1}^{p(n)} \int_{(R^{H(k)},m_k)} \left| Q_k(-nd_{k-1}) \right|^{2q(n)},$$

which is true because $|Q_k|$ is bounded by 1 for every k, and because each Q_k is a function of t_k only.

In order to estimate the factors on the right-hand side of (3–12), we study the trigonometric polynomials Q_k. For convenience we temporarily drop the indices k, n and consider a polynomial

$$Q(w) = H^{-1} \sum_{j=1}^{H} e(u_j w)$$

and even powers of its modulus

$$|Q(w)|^{2q} = Q(w)^q \overline{Q(w)}^q$$

$$= H^{-2q} \sum \frac{(q!)^2}{r_1! \cdots r_H!\, s_1! \cdots s_H!}\, e\left(\sum_{j=1}^{H} u_j(r_j - s_j) w \right)$$

where the summation is taken over all pairs

$$((r_1, \cdots, r_H),\ (s_1, \cdots, s_H))$$

of H-tuples of nonnegative integers adding up to q: $\sum_{j=1}^{H} r_j = q = \sum_{j=1}^{H} s_j$. This summation may be split into two parts Σ_1, Σ_2, the first sum being taken over the diagonal of the index set, i.e. the pairs of H-tuples with $r_j = s_j$ for $j = 1, \cdots, H$. Thus

$$|Q(w)|^{2q} = P_1 + P_2(w);$$

(3–13) $$P_1 = H^{-2q} \textstyle\sum_1 \dfrac{(q!)^2}{(r_1!\cdots r_H!)^2};$$

$$P_2(w) = H^{-2q} \textstyle\sum_2 \dfrac{(q!)^2}{r_1!\cdots r_H!\, s_1!\cdots s_H!} e\left(\sum_{j=1}^{H} u_j(r_j - s_j)w\right).$$

Let G be an integer $\geqq 2$ and let $q = GH$. If $r_1 + \cdots + r_H = q = GH$, then $(r_1!\cdots r_H!) \geqq (G!)^H$. Using Stirling's formula for the factorial,

$$n! = n^{n+(1/2)}e^{-n}c_n, \quad \frac{e}{\sqrt{2}} \leqq c_n \leqq e \quad (n = 1, 2, \cdots),$$

we obtain the estimate

$$P_1 \leqq H^{-GH}\frac{(GH)!}{(G!)^H} \leqq G^{(1/2)(1-H)}H^{1/2}e\, c_0^{-H}$$

where $c_0 = e/\sqrt{2} > 1$. Consequently,

(3–14) $$P_1 \leqq K \exp\left(-\frac{1}{2}(H-1)(\log G)\right)$$

where K is a constant independent of the choice of G and H. The quantity $P_2(w)$ is defined on (R^H, m) via the parametrization

$$u_j = \frac{3j-2}{3H} + \frac{t_j}{3Ha}, \quad j = 1, \cdots, H.$$

For any constant x,

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixt}\exp(-t^2/2)dt = \exp(-x^2/2);$$

in particular, assuming $r_j \neq s_j$,

$$\left|(2\pi)^{-1/2}\int_{-\infty}^{\infty} e(t_j(r_j - s_j)w/3Ha)\exp(-t_j^2/2)dt_j\right|$$

(3–15) $$= \exp\left\{-\frac{1}{2}[2\pi(r_j - s_j)w/3Ha]^2\right\}$$

$$\leqq \exp(-w^2/H^2a^2).$$

Since for every summand of $\sum_2$ in (3–13), $r_j - s_j \neq 0$ for at least one j, the expectation of $P_2(w)$ over (R^H, m) is dominated by the maximum of the quantities estimated in (3–15), i.e.,

(3–16) $$\int_{(R^H, m)} P_2(w) \leqq \exp(-w^2/H^2a^2).$$

From (3–14) and (3–16):

$$\int_{(R^H,m)} |Q(w)|^{2q} \leqq K\exp\left(-\frac{1}{2}(H-1)\log G\right) + \exp(-w^2/H^2a^2).$$

Let us now restore the indices k and n, and put $w = -nd_{k-1}$. We make G a function of k and assume for convenience that $G(k)$ and $H(k)$ are nondecreasing as $k \to \infty$. By the foregoing estimates and (3–12), letting $q(n) = G(p(n))H(p(n))$

$$\int_{(\mathcal{T},\nu)} |cc_n|^{2q(n)}$$

$$(3\text{–}17) \qquad \begin{aligned} &\leqq c^{2q(n)} \prod_{k=1}^{p(n)} \int_{(R^{H(k)},m_k)} |Q_k(-nd_{k-1})|^{2G(p(n))H(k)} \\ &\leqq c^{2q(n)} \prod_{k=1}^{p(n)} \Big\{K\exp\Big[-\frac{1}{2}(H(k)-1)\log G(p(n))\Big] \\ &\qquad\qquad + \exp[-n^2d_{k-1}^2/H(k)^2a_k^2]\Big\}. \end{aligned}$$

If n is large enough, to wit if for $p = p(n)$,

$$(3\text{–}18) \qquad n \geqq H(p)^{3/2}(\log G(p))^{1/2}a_pd_{p-1}^{-1},$$

then the last exponential in (3–17) is dominated by the first for each $k \leqq p(n)$, and

$$(3\text{–}19) \qquad \int_{(\mathcal{T},\nu)} |cc_n|^{2q(n)} \leqq c^{2q(n)}(K+1)^{p(n)}\exp\Big[-\frac{1}{2}(\log G(p(n)))\sum_{k=1}^{p(n)}(H(k)-1)\Big].$$

Let $Y(p)$ denote the right-hand side of (3–18). Let $p(n)$ be the largest integer p satisfying (3–18), i.e.

$$p(n) = \max\{p\colon n \geqq Y(p)\}, \qquad n = 1,2,\cdots;$$

$p(n)$ will have the same value r for fewer than $Y(r+1)$ consecutive values of n. By (3–19), (3–11) holds provided

$$\sum_{r=1}^{\infty} Y(r+1)c^{G(r)H(r)+r}\exp\Big[-\frac{1}{2}(\log G(r))\sum_{k=1}^{r}(H(k)-1)\Big] < \infty \text{ for all } c > 0.$$

Writing the expression for Y into this condition, we obtain

THEOREM C. *If a positive integer-valued function $G(k)$ can be defined such that*

$$(3\text{–}20) \qquad \begin{aligned} \sum_{r=1}^{\infty} d_r^{-1}\exp\Big[&\frac{3}{2}\log H(r+1) + \frac{1}{2}\log\log G(r+1) + \log a_{r+1} \\ &+ c(G(r)H(r)+r) - \frac{1}{2}(\log G(r))\sum_{k=1}^{r}(H(k)-1)\Big] \\ &< \infty \text{ for all } c > 0, \end{aligned}$$

then $E = E(\tau) = E\{(H(k), u_k(t_k), \xi_k)\}$ is an $\mathcal{M}_0$-set a.e. in $(\mathcal{E},\nu)$.

Clearly, (3–20) can be true, and the theorem useful, when the choice of the functions H and G is such that $(\log G(r)) \sum_{k=1}^{r}(H(k)-1)$ strongly dominates the other summands in the exponential bracket. In this event,

$$\beta < \frac{1}{2} \text{ and } \sum_{r=1}^{\infty} d_r^{-1}\exp(-\beta(\log G(r)) \sum_{k=1}^{\infty} (H(k)-1)) < \infty$$

suffices to give a family $\mathscr{E}$ made up almost entirely of $\mathscr{M}_0$-sets.

We consider two ways of defining the pair of functions G and H. First, let $H(k) \equiv 2$, $G(k) = k$. Then

$$(\log G(r)) \sum_{k=1}^{r} (H(k)-1) = r\log r;$$

and if $d_r = \exp(-\beta' r\log r)$ and $\beta' < \beta < 1/2$, then (3–20) is satisfied. Every set in the resulting family $\mathscr{E}$ is a symmetric set. For each k and each $\alpha > 0$, $J(k)d_k^\alpha = 2^k e^{-\alpha\beta' k\log k}$, which tends to zero as $k \to \infty$. We have established the following result of Salem ([5], p. 100):

Corollary C–1. *If $\beta < 1/2$, there exists a symmetric set whose Lebesgue measure is a pseudofunction, and such that*

(i) $\sum_{r=1}^{\infty} d_r^{-1}\exp(-\beta r\log r) < \infty$;

(ii) *the set has Hausdorff dimension zero, that is, it has zero h_α-measure for every $\alpha > 0$, where $h_\alpha(t) = t^\alpha$.*

This corollary gives a set satisfying (1–4) and thus complements Theorem A. The condition (*i*) cannot be much improved; for as Salem has shown ([8] or [5], p. 98), if $\lim_{r\to\infty} d_r^{1/r} r^{1/2} = 0$, then the Lebesgue measure on the set is *not* a pseudofunction.

Another alternative of interest is to take $H(k) = e^{k^\alpha}$, $\alpha < 1$; $G(k) \equiv 2$; $d_k = \exp(-e^{k^\alpha})$. Then (3–20) is satisfied, giving

Corollary C–2. *There exists a translation set which is an $\mathscr{M}_0$-set and has f_b-measure zero for all of the functions*

$$f_b(t) = \exp(-(\log t^{-1})^b), \qquad 0 < b < 1.$$

Completion of proof. To check the Hausdorff measures, note that $J(r) = \mathscr{O}(e^{r^{\alpha+1}})$ and $f_b(d_r) = \exp(-e^{br^\alpha})$, so that $J(r)f_b(d_r) \to 0$ as $r \to \infty$.

References

1. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press, London, 1957.

2. N. Dunford and J. T. Schwartz, *Linear operators, part* I: *General theory*, Interscience Publishers, New York, 1958.

3. O. S. Ivačev-Musatov (translated by Allen Shields), *M-sets and Hausdorff measure*, Soviet Math. Doklady. **3** (1962), 213–216.

4. ———, *The coefficients of trigonometric null series* (in Russian), Izvestiya Akad. Nauk SSSR, Sov. Mat. **21** (1957), 559–578.

5. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1963.

6. I. I. Pyateckii-Šapiro, *Supplement to the work*, "*On the problem of uniqueness of expansion of a function in a trigonometric series*" (in Russian), Učenye Zapiski Mosk. 165, Mat. **7** (1954), 78–97.

7. R. Salem, *Algebraic numbers and Fourier analysis*, D. C. Heath, Boston, London, 1963.

8. ———, *The absolute convergence of trigonometric series*, Duke Math. J. **8** (1941), 317–334.

9. A. Zygmund, *Trigonometric series*, I and II, Cambridge University Press, London, 1959.

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